

WEAK RATIONAL ERGODICITY DOES NOT IMPLY RATIONAL ERGODICITY

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ABSTRACT. We extend the notion of rational ergodicity to β -rational ergodicity for $\beta > 1$. Given $\beta \in \mathbb{R}$ such that $\beta > 1$, we construct an uncountable family of rank-one infinite measure preserving transformations that are weakly rationally ergodic, but are not β -rationally ergodic. The established notion of rational ergodicity corresponds to 2-rational ergodicity. Thus, this paper answers an open question by showing that weak rational ergodicity does not imply rational ergodicity.

1. INTRODUCTION

In this paper we consider ergodic properties of invertible, infinite measure-preserving transformations on σ -finite, nonatomic, Lebesgue measure spaces. As is well known, the averages in the ergodic theorem, for ergodic infinite measure-preserving transformations, converge to 0. In 1977, Aaronson [Aar77] introduced the notion of weak rational ergodicity, where an ergodic average for a certain class of sets converges to the expected limit, similar to the case of finite invariant measure. Aaronson also defined in the same article the notion of rational ergodicity and proved that rational ergodicity implies weak rational ergodicity but left the question of equivalence open. In this paper we define for each real number $\beta > 1$ a notion of β -rational ergodicity, with 2-rational ergodicity agreeing with the usual rational ergodicity. We then construct examples, for each $\beta > 1$, of β -rationally ergodic transformations which are not weakly rationally ergodic. Thus in particular we show that weak rational ergodicity does not imply rational ergodicity for infinite measure-preserving transformations.

2010 *Mathematics Subject Classification.* Primary 37A25; Secondary 28D05.

Key words and phrases. Ergodicity, Rational Ergodicity, Weakly Rationally Ergodic, Infinite Measure.

Let β be a real number and assume that $\beta > 1$. A transformation T is said to be **β -rationally ergodic** if it is conservative ergodic and there exists a set F of positive finite measure such that

$$\liminf_{n \rightarrow \infty} \frac{(\int_F \sum_{i=0}^{n-1} I_F(T^i x) d\mu)^\beta}{\int_F (\sum_{i=0}^{n-1} I_F(T^i x))^\beta d\mu} > 0.$$

The notion of rational ergodicity in [Aar77] corresponds to 2-rational ergodicity. A direct application of Hölder's inequality shows that if $\beta_2 > \beta_1 > 1$, and T is β_2 -rationally ergodic, then T is β_1 -rationally ergodic. Furthermore, T is said to be **weakly rationally ergodic** [Aar77] if it is conservative ergodic and there exists a set F of positive finite measure such that, if we set $a_n(F) = \sum_{k=0}^{n-1} \mu(F \cap T^k F) / \mu(F)^2$, then

$$\lim_{n \rightarrow \infty} \frac{1}{a_n(F)} \sum_{k=0}^{n-1} \mu(A \cap T^k B) = \mu(A)\mu(B),$$

for all measurable $A, B \subset F$.

2. CONSTRUCTION OF THE EXAMPLES

Let k_n, ℓ_n , and m_n be sequences of natural numbers.

2.1. Initialization. Let I_0 be an interval with positive length. Cut $C_0 = I_0$ into k_0 subintervals of equal length. Label the subintervals $C_0(i)$ for $0 \leq i < k_0$. Stack ℓ_0 subintervals on top of $C_0(i)$ for $0 \leq i < k_0 - 1$ to form $k_0 - 1$ subcolumns of height $\ell_0 + 1$. Label these subcolumns $\bar{C}_0(i)$ for $0 \leq i < k_0 - 1$. Stack the subcolumns $\bar{C}_0(i)$ for $0 \leq i < k_0 - 1$ from left to right to form a single subcolumn \bar{C}_0 of height $(k_0 - 1)(\ell_0 + 1)$. Let $\bar{C}_0(k_0 - 1) = C_0(k_0 - 1)$, which is a subcolumn of height 1. We have that both bases of towers \bar{C}_0 and $C_0(k_0 - 1)$ have the same measure:

$$\mu(C_0(k_0 - 1)) = \mu(C_0(0)) = \frac{1}{k_0} \mu(I_0).$$

Cut each subcolumn \bar{C}_0 and $C_0(k_0 - 1)$ into m_0 subcolumns and stack from left to right. In particular, let $C_0(i, j)$ be the j^{th} subcolumn of

$\bar{C}_0(i)$ for $0 \leq j < m_0$. Thus, as measurable sets,

$$\bar{C}_0 = \bigcup_{i=0}^{k_0-2} \bigcup_{j=0}^{m_0-1} C_0(i, j)$$

and

$$\bar{C}_0(k_0 - 1) = C_0(k_0 - 1) = \bigcup_{j=0}^{m_0-1} C_0(k_0 - 1, j).$$

Stack the $C_0(k_0 - 1)$ subcolumn of width $1/m_0 k_0$ on top of the \bar{C}_0 subcolumn of the same width to form a single column of height $m_0(k_0 - 1)(\ell_0 + 1) + m_0$. Place $m_0(k_0 - 1)(\ell_0 + 1) + m_0$ spacers on top to form column C_1 of height

$$h_1 = 2m_0(k_0 - 1)(\ell_0 + 1) + 2m_0.$$

2.2. General Step. Let C_n be a column of height h_n . Use the same procedure as above to cut C_n into k_n subcolumns of equal width. Separate the subcolumns into the first $k_n - 1$ subcolumns and the last subcolumn. Add ℓ_n subintervals on top of the first $k_n - 1$ subcolumns, and then stack from left to right to form a single subcolumn of height $(h_n + \ell_n)(k_n - 1)$. For the last subcolumn of height h_n , cut into m_n subcolumns of equal width and stack from left to right. Also, cut the first subcolumn of height $(h_n + \ell_n)(k_n - 1)$ into m_n subcolumns of equal width and stack from left to right. This produces two subcolumns of equal width. Stack the shorter subcolumn on top of the taller subcolumn, and add an equal number of spacers to form a single column C_{n+1} of height:

$$h_{n+1} = 2m_n(h_n + \ell_n)(k_n - 1) + 2m_n h_n.$$

Also, set $H_n = h_n + \ell_n$.

As in the initialization, let $C_n(i)$ be the i^{th} subcolumn from cutting C_n into k_n subcolumns of equal width. Let $\bar{C}_n(i)$ be the i^{th} subcolumn including the ℓ_n spacers added on top of $C_n(i)$ for $0 \leq i < k_n - 1$. Set $\bar{C}_n(k_n - 1) = C_n(k_n - 1)$. Finally, let $C_n(i, j)$ be the j^{th} subcolumn of $\bar{C}_n(i)$ for $0 \leq j < m_n$. For a given sequence $v = (v_n) = (k_n, \ell_n, m_n)$, this procedure produces a σ -finite measure preserving transformation $T_v : X \rightarrow X$ where $X = \bigcup_{n=1}^{\infty} C_n$.

Suppose L is the union of all subintervals added throughout the construction. Then $X \setminus L = I_{0,0}$ has finite μ measure, and the induced transformation $(T_v)_{X \setminus L}$ is ergodic and rank-one. For convenience, set $\mu(I_{0,0}) = 1$ and let \hat{T}_v denote the probability preserving invertible transformation obtained by inducing T_v on the set $X \setminus L$. Also, let \hat{h}_n be the tower height of the rank-one transformation \hat{T}_v corresponding to the tower of height h_n for T_v .

2.3. α -family. Given a real number x , let $\lfloor x \rfloor = \max \{\ell \in \mathbb{N} : \ell \leq x\}$. In this section, we restrict $v = (k_n, \ell_n, m_n)$ such that the collection of transformations T_v gives a sufficiently rich class of counterexamples. Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < 1$. Define the class V_α of infinite measure preserving transformations such that

$$V_\alpha = \{T_v : v = (n+1, \lfloor n^\alpha \rfloor h_n, m_n), \lim_{n \rightarrow \infty} \frac{\lfloor n^\alpha \rfloor}{m_n} = 0\}.$$

Define the collection

$$V = \bigcup_{0 < \alpha < 1} V_\alpha.$$

For $n \in \mathbb{N}$, $C_n(k_n - 1)$ is the last subcolumn of C_n . It is cut into m_n subcolumns of equal width, and labeled $C_n(k_n - 1, j)$ for $0 \leq j < m_n$.

Define

$$D_n = \bigcup_{j=\lfloor n^\alpha \rfloor}^{m_n-1} C_n(k_n - 1, j).$$

3. MAIN RESULTS

In this section, we state our main results, and give the proofs in the following two sections. The collection V provides all the necessary counterexamples, including a solution to the question of whether weak rational ergodicity implies rational ergodicity.

Theorem 3.1. *Each transformation $T \in V$ is a weakly rationally ergodic infinite measure preserving transformation.*

Theorem 3.2. *Suppose $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < 1$ and $\alpha\beta > 1$. If $T \in V_\alpha$, then for every set F of positive finite measure,*

$$\liminf_{n \rightarrow \infty} \frac{(\int_F \sum_{i=0}^{H_n-1} I_F(T^i x) d\mu)^\beta}{\int_F (\sum_{i=0}^{H_n-1} I_F(T^i x))^\beta d\mu} = 0.$$

In other words, T is not β -rationally ergodic.

Corollary 3.3. *For each $\beta > 1$, there exists an infinite measure preserving transformation T such that T is weakly rationally ergodic, but not β -rationally ergodic.*

Proof. Given $\beta > 1$, choose $\alpha < 1$ such that $\alpha\beta > 1$. Let T be any transformation in $V_\alpha \subset V$. By Theorem 3.1, T is weakly rationally ergodic, and by Theorem 3.2, T is not β -rationally ergodic. \square

Corollary 3.4. *There exist infinite measure preserving transformations T that are weakly rationally ergodic, but are not rationally ergodic.*

Proof. Apply Corollary 3.3 with $\beta = 2$. \square

4. WEAKLY RATIONALLY ERGODIC EXAMPLES

To establish weak rational ergodicity, we set $F = I_0$. Given $N \in \mathbb{N}$, define

$$a_N(\alpha) = \sum_{i=0}^{N-1} \mu(F \cap T_\alpha^i F).$$

Let $i, n \in \mathbb{N}$ be such that $0 \leq i \leq n$, and $F_n(i) = F \cap C_n(i)$. Define

$$b_N^n(\alpha) = \sum_{i=0}^{N-1} [\mu(F \cap T_\alpha^i F_n(n)) + \mu(F_n(n) \cap T_\alpha^i F)].$$

Lemma 4.1. *Suppose $t_n \in \mathbb{N}$ such that $0 < t_n < h_{n+1}$ for $n \in \mathbb{N}$. For $\alpha \in (0, 1)$,*

$$\lim_{n \rightarrow \infty} \frac{b_{t_n}^n(\alpha)}{a_{t_n}(\alpha)} = 0.$$

Proof. Let $T \in V_\alpha$, and $F_n(i, j) = F \cap C_n(i, j)$ for $n \in \mathbb{N}$. First, suppose $t_n < m_n h_n$. Let $p_n = \lfloor \frac{(n-1)m_n}{n} \rfloor$ and let

$$G_n = \bigcup_{i=0}^{k_n-2} \bigcup_{j=0}^{p_n} F_n(i, j).$$

Suppose $r \in \mathbb{N}$ such that $0 \leq r < m_n h_n - H_n$. Then for $0 \leq i < n$ and $0 \leq j < p_n$,

$$\sum_{t=0}^{H_n-1} \mu(G_n \cap T^{t+r} F_n(i, j)) = \sum_{t=0}^{h_n-1} \mu(F \cap T^{t+r} F_n(n, 0)).$$

Also, for $r \in \mathbb{N}$ such that $h_n \leq r < H_n$ and n sufficiently large,

$$\sum_{t=0}^{r-1} \mu(G_n \cap T^t F_n(i, j)) > \frac{1}{3} \sum_{t=0}^{h_n-1} \mu(F \cap T^t F_n(n, 0)).$$

Thus, for n sufficiently large,

$$\begin{aligned} \sum_{t=0}^{r-1} \mu(G_n \cap T^t G_n) &> \frac{p_n(k_n - 1)}{3} \sum_{t=0}^{h_n-1} \mu(F \cap T^t F_n(n, 0)) \\ &> \frac{m_n n}{6} \sum_{t=0}^{h_n-1} \mu(F \cap T^t F_n(n, 0)) \\ &\geq \frac{n}{6} \sum_{t=0}^{h_n-1} \mu(F \cap T^t F_n(n)) \\ &= \frac{n}{6(n^\alpha + 1)} (n^\alpha + 1) \sum_{t=0}^{h_n-1} \mu(F \cap T^t F_n(n)) \\ &\geq \frac{n}{6(n^\alpha + 1)} \sum_{t=0}^{r-1} \mu(F \cap T^t F_n(n)). \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \frac{6(n^\alpha + 1)}{n} = 0,$$

then our lemma holds for $h_n \leq t_n < H_n$. Similarly, it holds for $0 < t_n < H_n$. To establish for $H_n \leq t_n < m_n h_n$, let $t_n = q_n H_n + r_n$ where $q_n \in \mathbb{N}$ and $0 \leq r_n < H_n$. Then

$$\begin{aligned} (1) \quad \sum_{t=0}^{t_n-1} \mu(G_n \cap T^t G_n) &= \sum_{t=0}^{r_n-1} \mu(G_n \cap T^t G_n) \\ (2) \quad &+ \sum_{q=0}^{q_n-1} \sum_{t=0}^{H_n-1} \mu(G_n \cap T^{t+qH_n+r_n} G_n) \end{aligned}$$

We already established our lemma for (1), so we now handle (2).

$$\begin{aligned}
& \sum_{q=0}^{q_n-1} \sum_{t=0}^{H_n-1} \mu(G_n \cap T^{t+qH_n+r_n} G_n) \\
&= \sum_{q=0}^{q_n-1} \sum_{t=0}^{H_n-1} \sum_{i=0}^{k_n-2} \sum_{j=0}^{p_n-1} \mu(G_n \cap T^{t+qH_n+r_n} F_n(i, j)) \\
&= \sum_{q=0}^{q_n-1} \sum_{t=0}^{h_n-1} \sum_{i=0}^{n-1} \sum_{j=0}^{p_n-1} \mu(F \cap T^{t+qh_n+r_n} F_n(n, 0)) \\
&= \sum_{q=0}^{q_n-1} \sum_{t=0}^{h_n-1} np_n \mu(F \cap T^{t+qh_n+r_n} F_n(n, 0)) \\
&\geq \sum_{q=0}^{q_n-1} \sum_{t=0}^{h_n-1} \frac{nm_n}{2(n^\alpha + 1)} (n^\alpha + 1) \mu(F \cap T^{t+qh_n+r_n} F_n(n, 0)) \\
&\geq \sum_{q=0}^{q_n-1} \sum_{t=0}^{H_n-1} \frac{n}{2(n^\alpha + 1)} \mu(F \cap T^{t+qh_n+r_n} F_n(n)) \\
&= \sum_{t=r_n}^{t_n-1} \frac{n}{2(n^\alpha + 1)} \mu(F \cap T^t F_n(n))
\end{aligned}$$

Once again, since

$$\lim_{n \rightarrow \infty} \frac{2(n^\alpha + 1)}{n} = 0,$$

then our lemma is established for $0 < t_n < m_n h_n$.

Note that $\mu(F \cap T^t F_n(n)) = 0$ for $m_n h_n \leq t < h_{n+1} - m_n h_n$. If $t_n \geq h_{n+1} - m_n h_n$, the partial sum

$$\sum_{t=h_{n+1}-m_n h_n}^{t_n-1} \mu(F \cap T^t F_n(n))$$

may be handled in a similar manner as above. Also, the case of $\sum_{t=0}^{t_n-1} \mu(T^t F \cap F_n(n))$ follows in a similar way. This completes the proof of our lemma. \square

Lemma 4.2. *Let $T \in V_\alpha$ such that $0 < \alpha < 1$. Also, let $F = I_0$ and $A, B \subset F$ be measurable. Suppose $t_n = q_n H_n$ such that $1 \leq t_n < h_{n+1}$ for $n \in \mathbb{N}$. If*

$$a_{t_n} = \hat{h}_n q_n \left(1 - \frac{q_n}{2(n+1)m_n}\right),$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{a_{t_n}} \sum_{i=0}^{t_n-1} \mu(A \cap T^i B) = \mu(A)\mu(B).$$

Proof. This lemma may be proven using a counting argument on the $k_n m_n = (n+1)m_n$ subcolumns comprising C_n . By Lemma 4.1, we may assume $A \cap C_n(n) = \emptyset$ and $B \cap C_n(n) = \emptyset$. In this case, we may disregard $i \geq (n+1)m_n H_n$ in the summation, since $\mu(A \cap T^i B) = 0$ for $h_{n+1} > i \geq (n+1)m_n H_n$. In the above summation, each of the first $(nm_n - q_n)$ subcolumns, produces on average approximately weight

$$\frac{q_n \hat{h}_n \mu(A) \mu(B)}{(n+1)m_n}.$$

The next $(q_n - 1)$ subcolumns produces an approximate total weight

$$\frac{(q_n^2 - q_n)}{2(n+1)m_n} \hat{h}_n \mu(A) \mu(B).$$

Therefore, the total weight is approximately

$$\begin{aligned} & \hat{h}_n q_n \mu(A) \mu(B) \frac{(2nm_n - 2q_n + q_n - 1)}{2(n+1)m_n} \\ & \sim \hat{h}_n q_n \mu(A) \mu(B) \left(1 - \frac{q_n}{2(n+1)m_n}\right). \end{aligned}$$

□

The previous lemma gives a formula for a_t for certain values of $t \in \mathbb{N}$. Here we show how to define a_t for all t sufficiently large. Given $t \in \mathbb{N}$, choose $n \in \mathbb{N}$ such that $h_n \leq t < h_{n+1}$. Write $t = qH_n + r$ such that $0 \leq r < H_n$. To obtain the value of a_t , we separate into three cases based on the value of r :

- (1) $h_n \leq r < H_n - h_n$,
- (2) $r < h_n$,
- (3) $r \geq H_n - h_n$.

Case 1: Define a_t as

$$a_t = q \hat{h}_n \left(1 - \frac{q}{2(n+1)m_n}\right) + \frac{1}{2} \hat{h}_n.$$

Case 2: Let $r = q'H_{n-1} + r'$ where $0 \leq r' < H_{n-1}$. Define a_t as

$$a_t = q \hat{h}_n \left(1 - \frac{q}{2(n+1)m_n}\right) + q' \hat{h}_{n-1} \left(1 - \frac{q'}{2nm_{n-1}}\right).$$

Case 3: Let $H_n - r = q''H_{n-1} + r''$ where $0 \leq r'' < H_{n-1}$. Define a_t as

$$a_t = (q+1)\hat{h}_n(1 - \frac{q}{2(n+1)m_n}) - q''\hat{h}_{n-1}(1 - \frac{q''}{2nm_{n-1}})(1 - \frac{q}{(n+1)m_n}).$$

Theorem 4.3. Fix $\alpha \in (0, 1)$. Let $T \in V_\alpha$, $F = I_0$ and $A, B \subset F$ be measurable. Suppose a_t is defined as above for $t \in \mathbb{N}$. Then

$$\lim_{t \rightarrow \infty} \frac{1}{a_t} \sum_{i=0}^{t-1} \mu(A \cap T^i B) = \mu(A)\mu(B).$$

Proof. By passing to a subsequence, we may assume each of q, r, q', r', q'', r'' tends to ∞ or is bounded. For case 1, separate $a_t = b_t + c_t$ where $b_t = q\hat{h}_n(1 - \frac{q}{2(n+1)m_n})$ and $c_t = \frac{1}{2}\hat{h}_n$. Thus,

(3)

$$\begin{aligned} \frac{1}{a_t} \sum_{i=0}^{t-1} \mu(A \cap T^i B) = \\ (4) \quad \frac{b_t}{a_t} \frac{1}{b_t} \sum_{i=0}^{qH_n-1} \mu(A \cap T^i B) + \frac{c_t}{a_t} \frac{1}{c_t} \sum_{i=0}^{r-1} \mu(A \cap T^{qH_n+i} B) \end{aligned}$$

If $q = q(t) \rightarrow \infty$ as $t \rightarrow \infty$, then $c_t/a_t \rightarrow 0$ as $t \rightarrow \infty$, and we can disregard the second half of (4). In this case, our theorem follows from applying Lemma 4.2 to the first half of (4). Otherwise, the first half of (4) is approximated by

$$\frac{b_t}{a_t} \mu(A)\mu(B).$$

For case 1, most blocks of height h_n move forward into the spacers added to C_n under T^r . Since the blocks do not return to its neighboring block due to the spacers, then we get half of the intersection that would occur under $\hat{T}^{\hat{h}_n}$. Note, due to symmetry,

$$\sum_{i=0}^{r-1} \mu(A \cap T^{qH_n-i} B) \sim \sum_{i=0}^{r-1} \mu(A \cap T^{qH_n+i} B).$$

Thus, the second half of (4) is approximated by

$$\frac{c_t}{a_t} \mu(A)\mu(B).$$

Hence, for case 1,

$$\lim_{t \rightarrow \infty} \frac{1}{a_t} \sum_{i=0}^{t-1} \mu(A \cap T^i B) = \mu(A)\mu(B).$$

For case (2), if $r = 0$, then our theorem holds by Lemma 4.2. Likewise, if r is bounded, our theorem holds again by Lemma 4.2. If $c_t = q' \hat{h}_{n-1} (1 - \frac{q'}{2nm_{n-1}})$, and $r = q' H'_{n-1} + r'$, then q' bounded implies $c_t/a_t \rightarrow 0$. Otherwise,

$$(5) \quad \frac{1}{c_t} \sum_{i=0}^{r-1} \mu(A \cap T^i T^{qH_n} B)$$

$$(6) \quad = \frac{1}{c_t} \sum_{i=0}^{q' H_{n-1}-1} \mu(A \cap T^i T^{qH_n} B) + \frac{1}{c_t} \sum_{i=0}^{r'-1} \mu(A \cap T^i T^{q' H'_n} T^{qH_n} B).$$

By the previous argument, we can disregard the second half of (6), and hence,

$$(7) \quad \lim_{t \rightarrow \infty} \frac{1}{c_t} \sum_{i=0}^{r-1} \mu(A \cap T^i T^{qH_n} B) = \mu(A) \mu(B).$$

For case 3, let

$$\begin{aligned} \frac{1}{a_t} \sum_{i=0}^{t-1} \mu(A \cap T^i B) &= \frac{1}{(b_t - c_t)} \sum_{i=0}^{t-1} \mu(A \cap T^i B) = \\ &= \frac{b_t}{(b_t - c_t)} \frac{1}{b_t} \sum_{i=0}^{(q+1)H_n-1} \mu(A \cap T^i T^{qH_n} B) - \frac{c_t}{(b_t - c_t)} \frac{1}{c_t} \sum_{i=r}^{H_n-1} \mu(A \cap T^{qH_n+i} B) \end{aligned}$$

where $b_t = (q+1)\hat{h}_n(1 - \frac{q}{2(n+1)m_n})$ and $c_t = q''\hat{h}_{n-1}(1 - \frac{q''}{2nm_{n-1}})(1 - \frac{q}{(n+1)m_n})$. If $q \rightarrow \infty$ as $t \rightarrow \infty$, then $c_t/b_t \rightarrow 0$ as $t \rightarrow \infty$ and our result follows. Otherwise,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{c_t} \sum_{i=r}^{H_n-1} \mu(A \cap T^{qH_n+i} B) &= \lim_{t \rightarrow \infty} \frac{1}{c_t} \sum_{i=0}^{H_n-r-1} \mu(A \cap T^{(q+1)H_n-i-1} B) \\ &= \mu(A) \mu(B) \end{aligned}$$

and our proof is complete. \square

By setting $a_t(F) = a_t$, Theorem 4.3 clearly implies Theorem 3.1. Therefore, we have established that each $T \in V$ is weakly rationally ergodic.

5. NON-RATIONALLY ERGODIC EXAMPLES

Suppose $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < 1$ and $\alpha\beta > 1$. In this section, we prove for each $T \in V_\alpha$, T is not β -rationally ergodic. We note that there are many examples that have been shown to be rationally ergodic, see

e.g. [Aar97]. In particular, rank-one transformations with bounded cuts have been shown to be rationally ergodic [DGPS]. See also [AKW13, BSS⁺15]. Maharam transformations are not weakly rationally ergodic [Aar77], though they are not rank-one [BSS⁺15].

Before we prove the main theorem, we state and prove the following basic lemma.

Lemma 5.1. *Let T be an invertible infinite measure preserving ergodic transformation. Suppose for each set F of positive finite measure, there exists a sequence $t_n \in \mathbb{N}$ and $F_n \subset F$ of positive measure such that $\mu(F_n) \rightarrow 0$ as $n \rightarrow \infty$ and*

$$\limsup_{n \rightarrow \infty} \frac{\int_{F_n} \sum_{i=0}^{t_n-1} I_F(T^i x) d\mu}{\int_F \sum_{i=0}^{t_n-1} I_F(T^i x) d\mu} > 0.$$

Then T is not β -rationally ergodic for each $\beta > 1$.

Proof. Let $\beta > 1$ and $\gamma = \frac{\beta}{\beta-1}$. Without loss of generality, by passing to a subsequence, assume there exist $\eta > 0$ such that for all $n \in \mathbb{N}$,

$$\frac{\int_{F_n} \sum_{i=0}^{t_n-1} I_F(T^i x) d\mu}{\int_F \sum_{i=0}^{t_n-1} I_F(T^i x) d\mu} > \eta.$$

By Hölder's inequality,

$$\int_{F_n} \left(\sum_{i=0}^{t_n-1} I_F(T^i x) \right) I_{F_n}(x) d\mu \leq \left[\int_{F_n} \left(\sum_{i=0}^{t_n-1} I_F(T^i x) \right)^\beta d\mu \right]^{1/\beta} \mu(F_n)^{1/\gamma}$$

Thus,

$$\frac{[\int_{F_n} \sum_{i=0}^{t_n-1} I_F(T^i x) d\mu]^\beta}{\int_{F_n} (\sum_{i=0}^{t_n-1} I_F(T^i x))^\beta d\mu} \leq \mu(F_n)^{\beta/\gamma}$$

Therefore,

$$\begin{aligned} \frac{[\int_F \sum_{i=0}^{t_n-1} I_F(T^i x) d\mu]^\beta}{\int_F (\sum_{i=0}^{t_n-1} I_F(T^i x))^\beta d\mu} &< \left(\frac{1}{\eta}\right)^\beta \frac{[\int_{F_n} (\sum_{i=0}^{t_n-1} I_F(T^i x)) d\mu]^\beta}{\int_{F_n} (\sum_{i=0}^{t_n-1} I_F(T^i x))^\beta d\mu} \\ &\leq \left(\frac{1}{\eta}\right)^\beta \mu(F_n)^{\beta/\gamma} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. □

We will use the following lemma from [AS14]; we include the proof for completeness.

Lemma 5.2. (*Mixing Lemma*) *Let (X, γ) be a probability space. Let $E_i \subset X$ be a sequence of pairwise independent sets satisfying*

$$\sum_{i=1}^{\infty} \gamma(E_i) = \infty.$$

Given any measurable set $E \subset X$ and $\varepsilon > 0$, there exist infinitely many positive integers i such that $\gamma(E \cap E_i) > (\gamma(E) - \varepsilon)\gamma(E_i)$.

Proof: By squaring the integrand and applying independence, we get the following,

$$\int \left(\frac{1}{N} \sum_{i=1}^N (\chi_{E_i} - \gamma(E_i)) \right)^2 d\gamma = \frac{1}{N^2} \sum_{i=1}^N \gamma(E_i)(1 - \gamma(E_i)) < \frac{1}{N^2} \sum_{i=1}^N \gamma(E_i).$$

The Cauchy-Schwartz inequality implies

$$\begin{aligned} \left| \frac{1}{N} \sum_{i=1}^N (\gamma(E \cap E_i) - \gamma(E)\gamma(E_i)) \right| &= \left| \int_E \left(\frac{1}{N} \sum_{i=1}^N (\chi_{E_i} - \gamma(E_i)) \right) d\gamma \right| \\ &< \frac{1}{N} \sqrt{\sum_{i=1}^N \gamma(E_i)}. \end{aligned}$$

Thus,

$$\frac{\left| \sum_{i=1}^N (\gamma(E \cap E_i) - \gamma(E)\gamma(E_i)) \right|}{\sum_{i=1}^N \gamma(E_i)} < \frac{\sqrt{\sum_{i=1}^N \gamma(E_i)}}{\sum_{i=1}^N \gamma(E_i)} \rightarrow 0$$

as $N \rightarrow \infty$, since $\sum_{i=1}^{\infty} \gamma(E_i) = \infty$. Therefore, the lemma is established for every $\varepsilon > 0$. \square

Now we are ready for the proof of our second main theorem.

Proof of Theorem 3.2. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha < 1$ and $\alpha\beta > 1$. Let F be any set of positive finite measure. If we assume T is β -rationally ergodic, then, by Lemma 5.1, there exist $\delta_0 > 0$ and $n_0 \in \mathbb{N}$ such that if $F' \subset F$ satisfies $\mu(F') < \delta_0$, then for $t \geq n_0$,

$$\frac{\int_{F'} \sum_{i=0}^{t-1} I_F(T^i x) d\mu}{\int_F \sum_{i=0}^{t-1} I_F(T^i x) d\mu} < 1.$$

Let $\delta = \min \{\delta_0, 1/10\}$. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\delta}$ and there exists a union J of intervals in C_N such that

$$\frac{\mu(F \Delta J)}{\mu(J)} < 1 - \sqrt{1 - \delta^2}.$$

Let μ_N be normalized $\frac{\mu}{\mu(C_N)}$ probability measure on C_N . It is straightforward to see that the sets $C_N \cap C_n(k_n - 1)$ are independent for $n \geq N$ and $\sum_{n=N}^{\infty} \mu_N(C_N \cap C_n(k_n - 1)) = \infty$. Hence, by Lemma 5.2, there exists $n > N$ such that

$$\begin{aligned} \mu_N(F \cap J \cap C_N \cap C_n(k_n - 1)) & \\ & > \sqrt{1 - \delta^2} \mu_N(F \cap J) \mu_N(C_N \cap C_n(k_n - 1)) \\ & > (1 - \delta^2) \mu_N(J) \mu_N(C_N \cap C_n(k_n - 1)) \\ & = (1 - \delta^2) \mu_N(J \cap C_N \cap C_n(k_n - 1)) \end{aligned}$$

and such that both $H_n \geq n_0$ and

$$\frac{2^{2\beta} \mu(F)^{\beta-1}}{\frac{\lfloor n^\alpha \rfloor^\beta}{(n+1)} (1 - \delta)^2 (1 - 5\delta - \frac{2\lfloor n^\alpha \rfloor}{m_n})} < \delta.$$

The set $J \cap C_N \cap C_n(k_n - 1)$ is a union of subintervals in the sub-tower $C_n(k_n - 1)$. Suppose $\bar{J} = J \cap C_N \cap C_n(k_n - 1) = \bigcup_{i=0}^{p-1} J(i)$ where each $J(i)$ is a subinterval in $C_n(k_n - 1)$. Define

$$G = \{J(i) \subset J : \mu_N(J(i) \cap F) \geq (1 - \delta) \mu(J(i))\}.$$

For convenience, associate $G = \bigcup_{J(i) \in G} J(i)$. Then $\mu_N(G) > (1 - \delta) \mu_N(\bar{J})$. If $q \in \mathbb{N}$ such that $0 \leq q < \lfloor n^\alpha \rfloor$, then for $J_j, J_k \in G$,

$$\mu((F \cap J_j) \cap (\bigcup_{i=0}^{h_n-1} T^{-qh_n-i}(F \cap J_k))) > (1 - 2\delta - \frac{\lfloor n^\alpha \rfloor}{m_n}) \mu(J_j).$$

Thus, there exists a subset $J_j^* \subset J_j$ satisfying

$$\mu(J_j^*) > (1 - 4\delta - \frac{2\lfloor n^\alpha \rfloor}{m_n}) \mu(J_j)$$

such that for $x \in J_j^*$,

$$\sum_{J_k \in G} \sum_{q=0}^{\lfloor n^\alpha \rfloor - 1} \sum_{i=0}^{h_n-1} I_{F \cap J_k}(T^{qh_n+i}x) > \frac{p\lfloor n^\alpha \rfloor}{2}.$$

Hence,

$$\begin{aligned} \int_{F \cap J_j} \left(\sum_{J_k \in G} \sum_{q=0}^{\lfloor n^\alpha \rfloor - 1} \sum_{i=0}^{h_n - 1} I_{F \cap J_k}(T^{qh_n + i}x) \right)^\beta d\mu \\ > \left(\frac{p \lfloor n^\alpha \rfloor}{2} \right)^\beta (1 - 5\delta - \frac{2 \lfloor n^\alpha \rfloor}{m_n}) \mu(J_j) \end{aligned}$$

This implies

$$(8) \quad \int_F \left(\sum_{i=0}^{H_n - 1} I_F(T^i x) \right)^\beta d\mu \geq \sum_{J_j \in G} \int_{F \cap J_j} \left(\sum_{J_k \in G} \sum_{i=0}^{H_n - 1} I_{F \cap J_k}(T^i x) \right)^\beta d\mu$$

$$(9) \quad > \left(\frac{p \lfloor n^\alpha \rfloor}{2} \right)^\beta (1 - 5\delta - \frac{2 \lfloor n^\alpha \rfloor}{m_n}) \mu(G)$$

$$(10) \quad > \left(\frac{p \lfloor n^\alpha \rfloor}{2} \right)^\beta (1 - 5\delta - \frac{2 \lfloor n^\alpha \rfloor}{m_n}) (1 - \delta) \frac{\mu(J)}{(n+1)}$$

$$(11) \quad > \frac{(1 - \delta)^2}{(n+1)} \left(\frac{p \lfloor n^\alpha \rfloor}{2} \right)^\beta (1 - 5\delta - \frac{2 \lfloor n^\alpha \rfloor}{m_n}) \mu(F).$$

Let $\hat{J} = J \cap C_N \setminus C_n(k_n - 1)$ and $\bigcup_{i=0}^{p-1} \hat{J}_i = \hat{J}$ where each \hat{J}_i is a subinterval in $C_n \setminus C_n(k_n - 1)$. We have

$$(12) \quad \int_{F \cap \hat{J}} \sum_{i=0}^{H_n - 1} I_F(T^i x) d\mu \leq \int_{\hat{J}} \sum_{i=0}^{H_n - 1} I_F(T^i x) d\mu$$

$$(13) \quad = \sum_{j=0}^{p-1} \sum_{i=0}^{H_n - 1} \int_{T^{-i} \hat{J}_j} I_F(x) d\mu$$

$$(14) \quad \leq \sum_{j=0}^{p-1} \mu(F) = p\mu(F).$$

Since $\mu(F \setminus \hat{J}) < \delta \leq \delta_0$, then

$$\int_F \sum_{i=0}^{H_n - 1} I_F(T^i x) d\mu \leq 2p\mu(F).$$

Therefore,

$$\begin{aligned} \frac{(\int_F \sum_{i=0}^{H_n - 1} I_F(T^i x) d\mu)^\beta}{\int_F (\sum_{i=0}^{H_n - 1} I_F(T^i x))^\beta d\mu} &< \frac{2^\beta p^\beta \mu(F)^\beta}{\frac{(1-\delta)^2}{(n+1)} \left(\frac{p \lfloor n^\alpha \rfloor}{2} \right)^\beta (1 - 5\delta - \frac{2 \lfloor n^\alpha \rfloor}{m_n}) \mu(F)} \\ &= \frac{2^{2\beta} \mu(F)^{\beta-1}}{\frac{\lfloor n^\alpha \rfloor^\beta}{(n+1)} (1 - \delta)^2 (1 - 5\delta - \frac{2 \lfloor n^\alpha \rfloor}{m_n})} < \delta. \end{aligned}$$

Since $\delta > 0$ may be chosen arbitrarily small, this contradicts the assumption that T is β -rationally ergodic and completes the proof of our theorem. \square

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